S520

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1 4.3 Expectation Examples

1. Definition: The expected value of a discrete rv is the probability-weighted average of it's possible values.

x	$f(x)$	$xf(x)$
-1	0.1	-0.1
0	0.2	0
1	0.1	0.1
4	0.6	2.4
		$E(X) = 2.4$

Notation:

$$E[X] = EX = \sum_{x \in X(S)} x(P(X = x)) = \sum_{x \in X(S)} xf(x)$$

Often we write $\boxed{\text{EX} = \mu}$—the "population mean".

Example: rv $Y \sim$ Geometric (p)

$$\begin{aligned} Y(S) &= \{0, 1, 2, \ldots\} \\ f(k) &= (1-p)^k p \\ EY &= \sum_{k=0}^{\infty} k(f(x)) = \sum_{k=0}^{\infty} k(1-p)^k p \\ &= \text{Calculus magic we don't care about} \quad (1) \\ &= \frac{1}{p} - 1 \quad (2) \end{aligned}$$

If $p = 0.01$, $EY = \frac{1}{0.01} - 1 = 100 - 1 = 99$

2. Properties of Expected Value

   (a) If a random variable is always a given constant, the expected value is that constant.

   $$X \equiv c \Rightarrow EX = c$$

   (b) Taking the expected value of an rv plus a constant is the same as taking the expected value of the rv then adding that constant.

   $$E(X + b) = EX + b$$

(c) Taking the expected value of an rv multiplied by a constant is the same as taking the expected value of the rv then multiplying by that constant.

$$E(cX) = cEX$$

(d) The expected value of an rv plus another rv is the expected value of the first rv plus the expected value of the second rv.

$$E(X + Y) = EX + EY$$

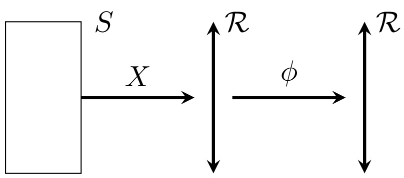

Figure 1: Mapping an rv X onto an rv ϕ.

3. $Y = \phi(X)$ is a random variable

$$EY = E\phi(X) = \sum_{x \in X(S)} \phi(x) f(x)$$

Experiment 1		Experiment 2	
x	$f(x)$	x	$f(x)$
4.8	0.2	0	0.5
5.0	0.6	100	0.5
5.2	0.2		
$\mu_1 = 5$		$\mu_2 = 5$	

It is often important to understand how far away the values are from a given value. $f(x) = x^2$ often leads to "beautiful" mathematical properties.

- Consider $\phi(x) = (x - \mu)^2$ where $\mu = EX$.

Definition: The variance of a discrete random variable is the probability-weighted average of the squared deviations of its values from the population mean.

Notation:

$$Var(X) = VarX = E(X - \mu)^2 = \sum_{x \in X(S)} (x - \mu)^2 f(x)$$

We often write $VarX = \sigma^2$, the population variance. - Squared because : it's always positive and (more importantly) $\sqrt{\sigma^2} = \sigma$ is the population standard deviation.

x	$f(x)$	$xf(x)$	$(x-\mu)^2$	$(x-\mu)^2 f(x)$
-1	0.1	-0.1	11.56	1.156
0	0.2	0.0	5.76	1.152
1	0.1	0.1	1.96	0.196
4	0.6	2.4	2.56	1.536
				$4.040 = VarX = \sigma^2$

Figure 2: The first table from the beginning of class. The problems in the homework are basically this table.

4. (a) If an rv is always equal to a constant, then the variance is 0.

$$X \equiv c \Rightarrow VarX = 0$$

(b) Adding a constant to an rv does not affect the variance of that rv.

$$Var(X+b) = VarX$$

(c) Taking the variance of an rv multiplied by a constant is the same as multiplying the variance of the rv by the constant squared. (Because the variance is σ^2 and some extra math I didn't write down.)

$$Var(cX) = c^2 VarX$$

(d) ıIf and only if X and Y are independent, then taking the variance of the sum of X and Y is the same as summing the variance of each rv separately.

$$Var(X+Y) = VarX + VarY$$

(e)

$$\begin{aligned} Var(X) &= E(X-\mu)^2 \\ &= E\left[x^2 - 2\mu x + \mu^2\right] \\ &= E\left[x^2\right] + E\left[-2\mu x\right] + E\left[mu^2\right] \\ &= E\left[x^2\right] - 2\mu E\left[x\right] + mu^2 \\ &= EX^2 - 2\mu^2 + mu^2 \\ &= EX^2 - \mu^2 \\ &= \boxed{E(X^2) - (EX)^2} \Leftarrow \text{this is often less work than the other way} \end{aligned}$$

5. On Monday, Binomial Experiments