S520

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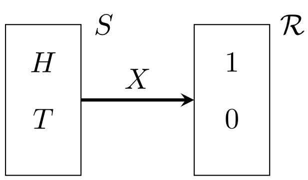
1 4. Discrete Random Variables

1. We'll start working with R in this chapter, but not use R heavily until Chapter 7.
2. Definition: X is a discrete rv iff $X(S)$ is countable.

   Example: (Trivial, but foundational)



$\Rightarrow X(S) = 0, 1$

Figure 1: Coin toss mapping.

   Remark: cdf F of a descrete rv X is a step function with jumps precisely at the values of $X(S)$ $[x \in X(S)]$.

   If we have an X that is a possible value, the jump in F at $x = P(X = x)$

3. Definition: Let X denote a descrete random variable. The probability mass function (pmf) of X is the function $f : \mathcal{R} \rightarrow \mathcal{R}$ defined by:

$$f(x) = \begin{cases} P(x) & \text{, if } x \in X(S) \\ 0 & \text{, if } x \notin X(S) \end{cases}$$

   What's $P(Y \in (0.6, 3.6))$?

$$\begin{aligned} P(Y \in (0.6, 3.6) &= P(\{1, 2\})) \\ &= P(Y = 1) + P(Y = 2) \\ &= f(1) + f(2) \\ &= \frac{1}{2} + \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

   Back to the previous example:

   P(Heads) = p   P(Tails) = 1 - p
   || ||
   P(X = 1)   P(X = 0)
   || ||
   f(1)   f(0)
   Successes   Failures

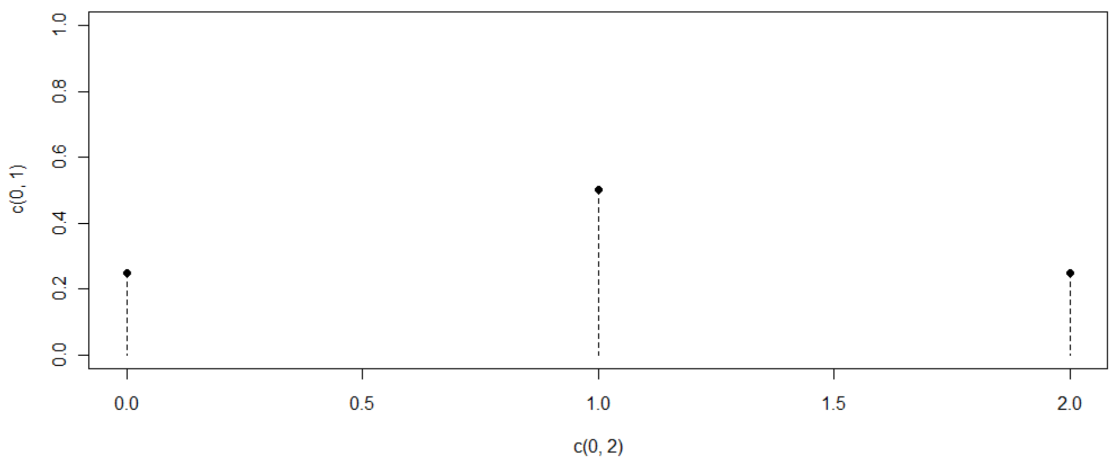


Figure 2: pmf of two coin tosses

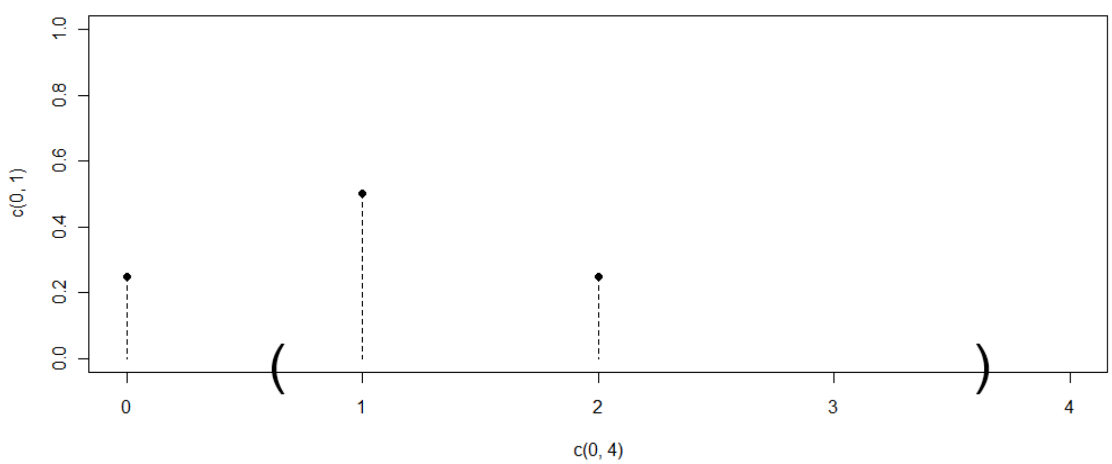


Figure 3: What's $P(Y \in (0.6, 3.6))$?

Everything is defined in terms of $P(Heads)$, so it's considered the parameter.

When we have only two possible solutions, we have what is called a Bernoulli trial.

We show this by $X \sim Bernoulli(p)$.

4. A more complicated example using Bernoulli trials. Example:

   A city builds a system of levees for hurricane protection. Suppose that the probability that a levee will break in a category 3 hurricane is 1%.

   (a) Question: What's $P(\text{levee surviving 10 Category 3 hurricanes})$?

   Answer: Think of a Cat-3 hurricane i is a Bernoulli trial. Then:

$$X_i = \begin{cases} 1, \text{if levees breach} \\ 0, \text{if levees survive} \end{cases} \sim \text{Bernoulli}(p = 0.01)$$

Assume X_i is independent, i.e., that the levees are restored to their former condition after each hurricane. Then:

$$\begin{aligned} P(\text{levees survived 10 consecutive Cat-3 hurricanes}) &= \\ &= P(X_1 = 0, X_2 = 0, \ldots, X_10 = 0) \\ &= P(X_1 = 0) \times P(X_2 = 0) \times \cdots \times P(X_10 = 0) \\ &= 0.99^{10} \approx 90\% \end{aligned}$$

(b) Question: How many Cat-3 hurricanes should we <u>expect</u> the levees to survive?

Answer: Define Y = number of hurricanes we survive before the first breach. Then $Y(S) = 0, 1, 2, \ldots$ and for $k \in Y(S)$:

$$\begin{aligned} f(k) &= P(Y = k) = P(X_1 = 0, X_2 = 0, \ldots, X_k = 0) \\ &= P(X_1 = 0) \times P(X_2 = 0) \times \cdots \times P(X_k = 0) \\ &= (1 - p)^k p \end{aligned}$$

So, Y is a geometric random variable. That means that $Y \sim \text{Geometric}(p)$.

5. Expected value (Section 4.3 for Friday):

Imagine a simulation experiment performed with a coin for which $P(Heads) = 0.01$. We toss the coin until we get $Heads$, and count Y = number of $Tails$.

We observe:

$$\begin{aligned} y &= 43 \\ y &= 17 \\ y &= 209 \end{aligned}$$

What number of $Heads$ do we expect to get under normal conditions?