S520

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1 3.5. Random Variables

1. A random variable (RV) is a rule or function that assigns a real number to each experimental outcome.
2. Intuitively, this means:
   (a) Example 1: Outcome = Leone Film (problem from the first homework) Label = Year of release
   (b) Example 2: Flipping a coin (or a coin toss) It can be helpful to replace the normal outcomes with numbers. General notation—we usually denote RV's by letters like X, Y, and Z

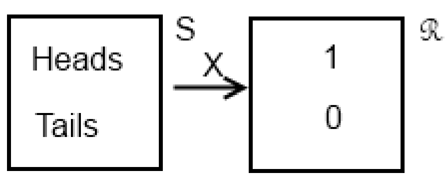

Figure 1: Coin Toss rv.

   i. Can use similar notation to $f(x) = x^2$ and $f(5) = 25$

$$X(Heads) = 1$$

   ii. $X(S)$ = range of X, i.e., the set of real numbers that are assigned to experimental outcomes

$$X(S) = 0, 1$$

3. Cumulative Distribution Function (CDF)
   (a) A CDF F of an RV X is the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(y) = P(X \leq y)$.
   (b) Back to example 2: A Fair Coin Toss

$$P(H) = \frac{1}{2}, P(T) = \frac{1}{2}$$

I want $y = \frac{1}{3}$, so $F(\frac{1}{3}) = P(X \leq \frac{1}{3})$. $\Rightarrow$ Use the 1, 3 notation, so $F(\frac{1}{3}) = P(X \leq \frac{1}{3}) = \frac{1}{2}$, since 0 is the only value in my outcomes less than $\frac{1}{3}$. Explanation of notation:

$$\begin{aligned} F(y) &= P(X \leq y) \\ &= P(\{s \in S : X(s) \leq y\}) \end{aligned}$$

$$X^{-1}((-\infty, y]) = \text{inverse image of } X$$

CDF's only make sense when the sets in question have/are events, which is the other condition on random variables. –Technical aside: anything in this course will be fine to work with.

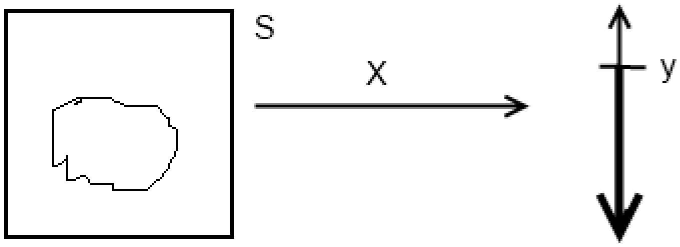

Figure 2: Mapping from an experiment to an RV.

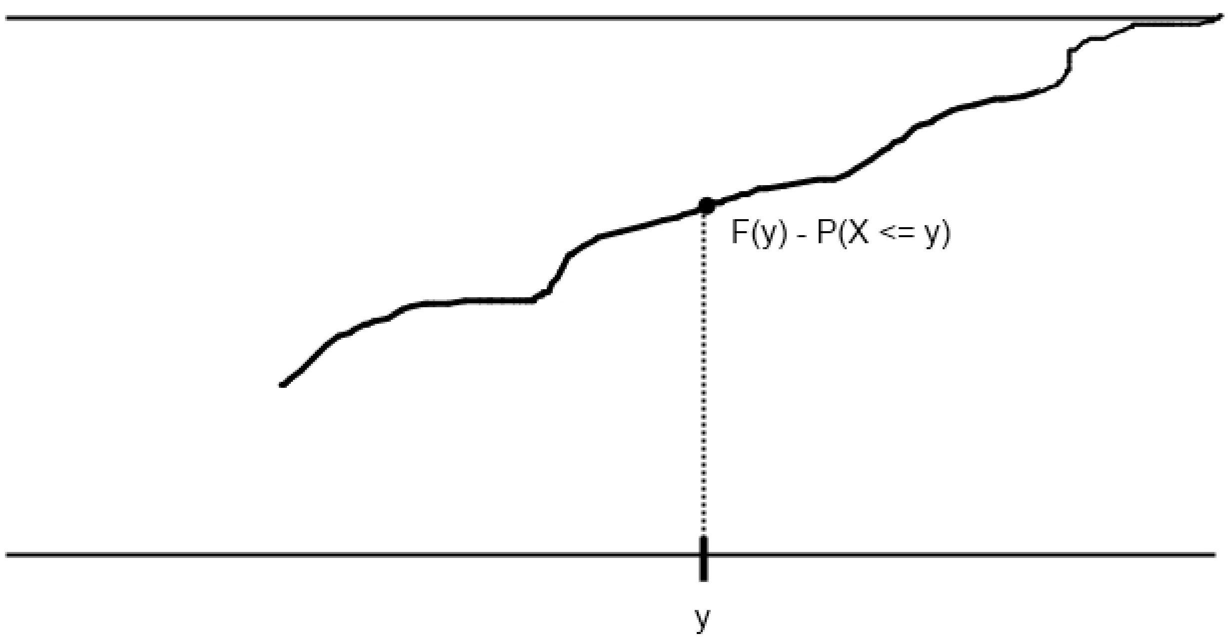

Figure 3: cdfs are non-decreasing

4. Working with CDF's

   (a) Note 1: $F(y)$ is a probability. Therefore:

$$0 \leq F(y) \leq 1 \text{ for every } y \in (-\infty, \infty)$$

   (b) Note 2:

$$\begin{aligned} F(10) &\leq F(20) \\ F(x) &\leq F(y) \text{ if } x \leq y \end{aligned}$$

$$A = X \leq 10 \subset x \leq 20$$

By the property of probability:

$$P(A) \leq P(B)$$

Point of the matter: CDF's are non-decreasing.

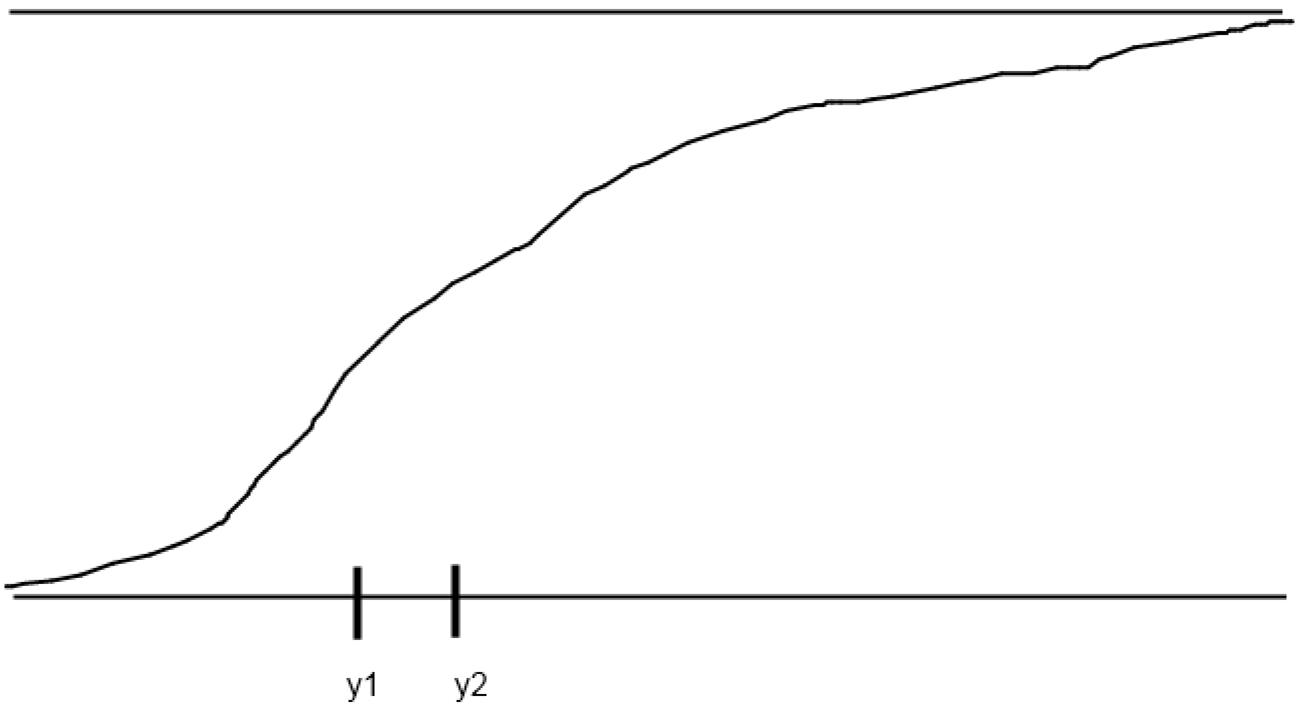

Figure 4: Comparisons of points of a cdf

(a) $P(X > y_2) = 1 - P(X \leq y_2) = 1 - F(y_2)$
If we don't have an $X \leq x$ probability, we can use previous properties to figure things out. We often have to massage things to get them top work.

(b) $P(y_1) < X \leq y_2) = P(X \leq y_2) - P(X \leq y_1) = F(y_2) - F(y_1)$

(c) Example from Book : toss 2 fair coins, count # of Heads $F(-2) = P(Y \leq 2) = 0$ $F(0) = P(Y \leq 0) = 0.25$

5. Preview of Chap. 4 - Discrete RV's

   (a) Properties

      i. CDF's are step functions

         A. Chapter 5 - Continuous RV's

      ii. Discrete random variables illustrate some basic principles appropriate to all RV's that are easeier to understand here than in continuous RV's.

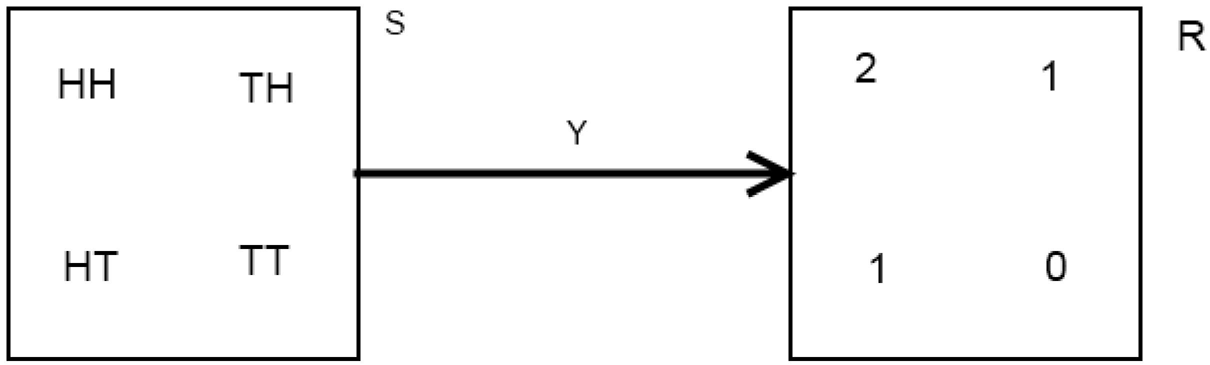

Figure 5: Sample space and mapping of two fair coin tosses.

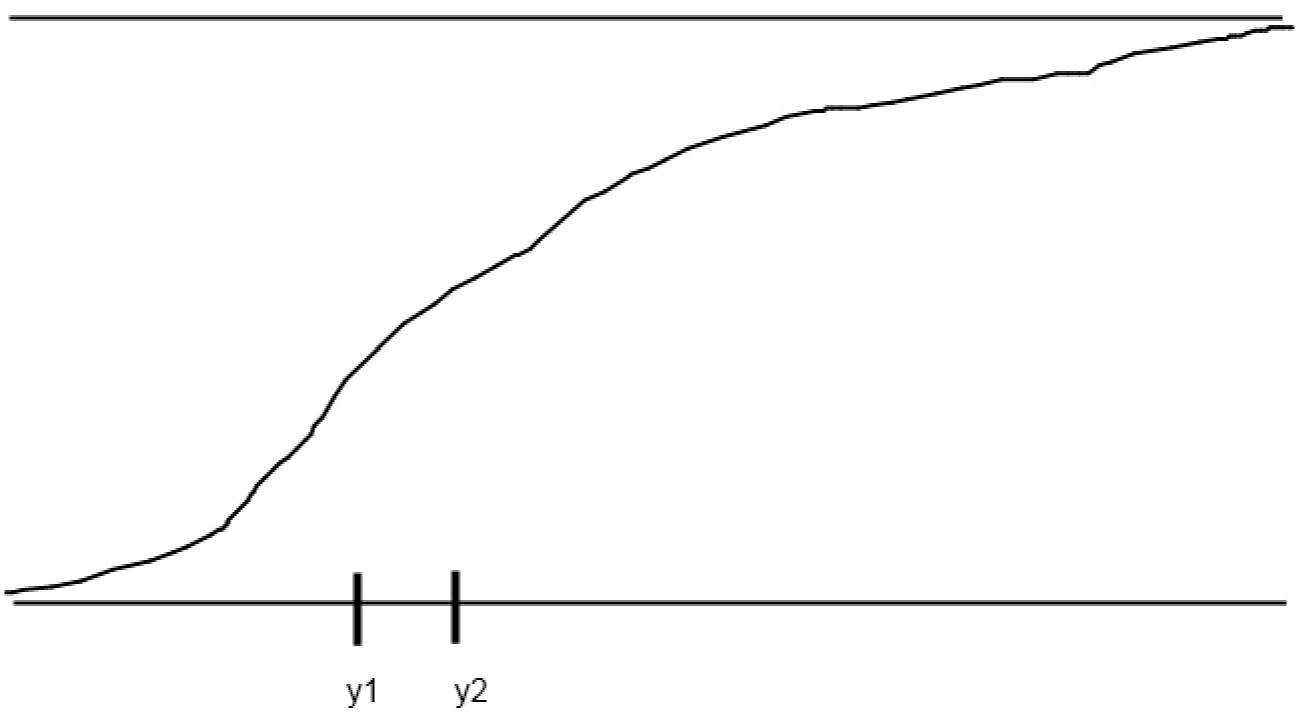

Figure 6: CDF for two fair coin tosses.